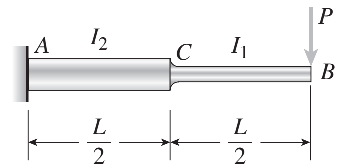
Nonprismatic Beams

Problem 9.7-1 The cantilever beam *ACB* shown in the figure has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the load *P*.

(b) Determine the ratio *r* of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.

(c) Plot a graph of the deflection ratio *r* versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

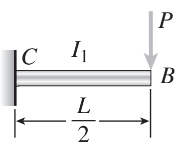


Solution 9.7-1 **Cantilever beam (nonprismatic)**

Use the method of superposition.

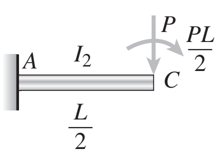
(a) DEFLECTION δ_B AT THE FREE END

(1) Part *CB* of the beam:



$$(\delta_B)_1 = \frac{P}{3EI_1}\left(\frac{L}{2}\right)^3 = \frac{PL^3}{24EI_1}$$

(2) Part *AC* of the beam:



$$\delta_C = \frac{P(L/2)^3}{3EI_2} + \frac{(PL/2)(L/2)^2}{2EI_2} = \frac{5PL^3}{48EI_2}$$

$$\theta_C = \frac{P(L/2)^2}{2EI_2} + \frac{(PL/2)(L/2)}{EI_2} = \frac{3PL^2}{8EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{7PL^3}{24EI_2}$$

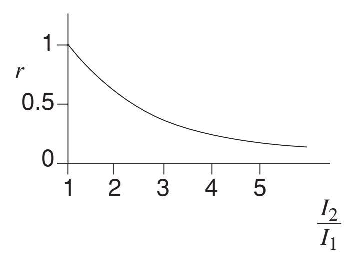
(3) Total deflection at point *B*

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{PL^3}{24EI_1}\left(1 + \frac{7I_1}{I_2}\right) \quad \longleftarrow$$

(b) PRISMATIC BEAM $\quad \delta_1 = \frac{PL^3}{3EI_1}$

Ratio: $\quad r = \frac{\delta_B}{\delta_1} = \frac{1}{8}\left(1 + \frac{7I_1}{I_2}\right) \quad \longleftarrow$

(c) GRAPH OF RATIO



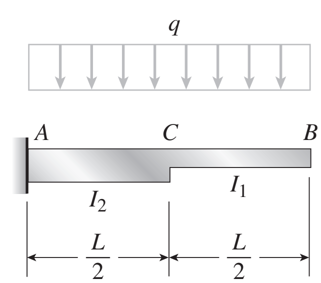
$\frac{I_2}{I_1}$	r
1	1.00
2	0.56
3	0.42
4	0.34
5	0.30

Problem 9.7-2 The cantilever beam *ACB* shown in the figure supports a uniform load of intensity *q* throughout its length. The beam has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the uniform load.

(b) Determine the ratio *r* of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.

(c) Plot a graph of the deflection ratio *r* versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

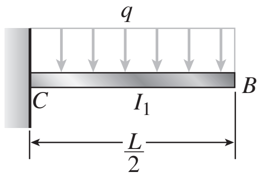


Solution 9.7-2 Cantilever beam (nonprismatic)

Use the method of superposition

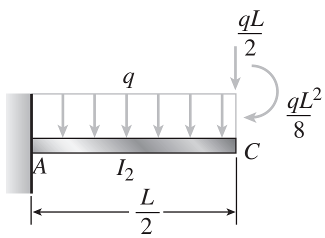
(a) DEFLECTION δ_B AT THE FREE END

(1) Part CB of the beam:



$$(\delta_B)_1 = \frac{q}{8EI_1}\left(\frac{L}{2}\right)^4 = \frac{qL^4}{128EI_1}$$

(2) Part AC of the beam:



$$\delta_C = \frac{q(L/2)^4}{8EI_2} + \frac{\left(\frac{qL}{2}\right)(L/2)^3}{3EI_2} + \frac{\left(\frac{qL^2}{8}\right)\left(\frac{L}{2}\right)^2}{2EI_2} = \frac{17qL^4}{384EI_2}$$

$$\theta_C = \frac{q(L/2)^3}{6EI_2} + \frac{(qL/2)(L/2)^2}{2EI_2} + \frac{(qL^2/8)(L/2)}{EI_2}$$

$$= \frac{7qL^3}{48EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{15qL^4}{128EI_2}$$

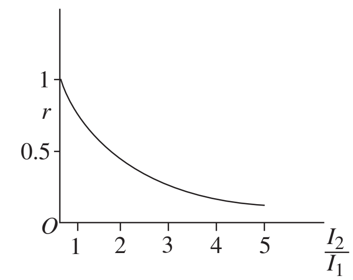
(3) Total deflection at point B

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qL^4}{128EI_1}\left(1 + \frac{15I_1}{I_2}\right) \quad \longleftarrow$$

(b) PRISMATIC BEAM $\quad \delta_1 = \frac{qL^4}{8EI_1}$

Ratio: $\quad r = \frac{\delta_B}{\delta_1} = \frac{1}{16}\left(1 + \frac{15I_1}{I_2}\right) \quad \longleftarrow$

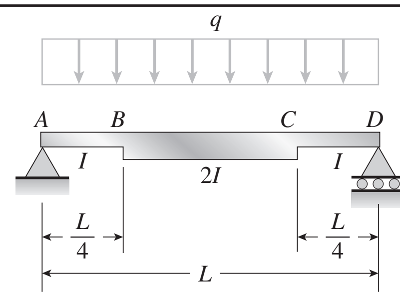
(c) GRAPH OF RATIO



$\frac{I_2}{I_1}$	r
1	1.00
2	0.53
3	0.38
4	0.30
5	0.25

Problem 9.7-3 A simple beam $ABCD$ has moment of inertia I near the supports and moment of inertia $2I$ in the middle region, as shown in the figure. A uniform load of intensity q acts over the entire length of the beam.

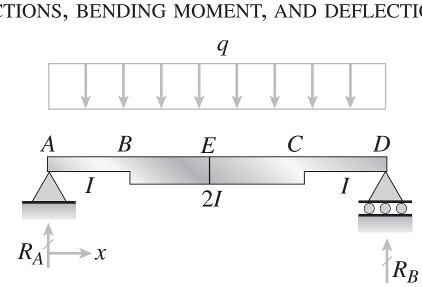
Determine the equations of the deflection curve for the left-hand half of the beam. Also, find the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.



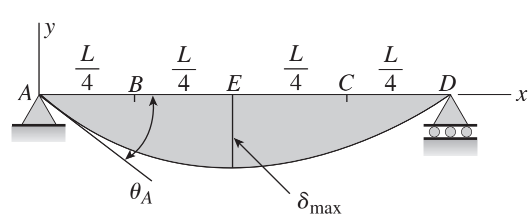
Solution 9.7-3 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-12a).

REACTIONS, BENDING MOMENT, AND DEFLECTION CURVE



$$R_A = R_B = \frac{qL}{2} \qquad M = Rx - \frac{qx^2}{2} = \frac{qLx}{2} - \frac{qx^2}{2}$$



(Continued)

BENDING-MOMENT EQUATIONS FOR THE LEFT-HAND HALF OF THE BEAM

$$EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{4}\right) \qquad (1)$$

$$E(2I)v'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \qquad (2)$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{4}\right) \qquad (3)$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_2 \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \qquad (4)$$

B.C. 1 Symmetry: $v'\left(\frac{L}{2}\right) = 0$

From Eq. (4): $C_2 = -\frac{qL^3}{24}$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \qquad (5)$$

SLOPE AT POINT B (FROM THE RIGHT)

Substitute $x = \frac{L}{4}$ into Eq. (5):

$$EIv_B' = -\frac{11\,qL^3}{768} \qquad (6)$$

B.C. 2 CONTINUITY OF SLOPES AT POINT B

$(v_B')_{\text{Left}} = (v_B')_{\text{Right}}$

From Eqs. (3) and (6):

$$\frac{qL}{4}\left(\frac{L}{4}\right)^2 - \frac{q}{6}\left(\frac{L}{4}\right)^3 + C_1 = -\frac{11qL^3}{768} \qquad \therefore\ C_1 = -\frac{7qL^3}{256}$$

SLOPES OF THE BEAM (from Eqs. 3 and 5)

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{7qL^3}{256} \quad \left(0 \le x \le \frac{L}{4}\right) \qquad (7)$$

$$EIv' = \frac{qLx^2}{8} - \frac{qx^3}{12} - \frac{qL^3}{48} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \qquad (8)$$

ANGLE OF ROTATION θ_A (FROM EQ. 7)

$$\theta_A = -v'(0) = \frac{7qL^3}{256EI} \text{ (positive clockwise)} \quad \longleftarrow$$

INTEGRATE EQS. (7) AND (8)

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{7qL^3x}{256} + C_3 \quad \left(0 \le x \le \frac{L}{4}\right) \qquad (9)$$

$$EIv = \frac{qLx^3}{24} - \frac{qx^4}{48} - \frac{qL^3x}{48} + C_4 \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \qquad (10)$$

B.C. 3 Deflection at support A

$v(0) = 0$ From Eq. (9): $C_3 = 0$

DEFLECTION AT POINT B (FROM THE LEFT)

Substitute $x = \frac{L}{4}$ into Eq. (9) with $C_3 = 0$:

$$EIv_B = -\frac{35\,qL^4}{6144} \qquad (11)$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Right}} = (v_B)_{\text{Left}}$

From Eqs. (10) and (11):

$$\frac{qL}{24}\left(\frac{L}{4}\right)^3 - \frac{q}{48}\left(\frac{L}{4}\right)^4 - \frac{qL^3}{48}\left(\frac{L}{4}\right) + C_4 = -\frac{35qL^4}{6144}$$

$$\therefore\ C_4 = -\frac{13qL^4}{12{,}288}$$

DEFLECTIONS OF THE BEAM (FROM EQS. 9 AND 10)

$$v = -\frac{qx}{768EI}(21L^3 - 64Lx^2 + 32x^3) \quad \left(0 \le x \le \frac{L}{4}\right) \quad \longleftarrow$$

$$v = -\frac{q}{12{,}288EI}(13L^4 + 256L^3x - 512Lx^3 + 256x^4) \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \quad \longleftarrow$$

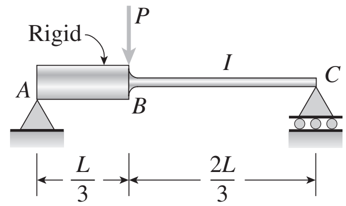
MAXIMUM DEFLECTION (AT THE MIDPOINT E)

(From the preceding equation for v.)

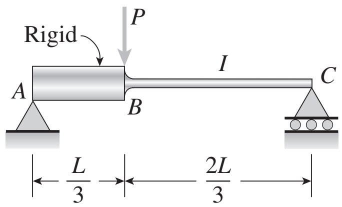
$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{31qL^4}{4096EI} \text{ (positive downward)} \quad \longleftarrow$$

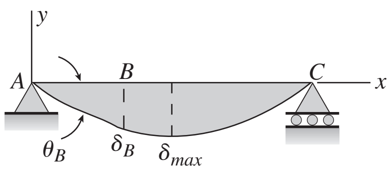
Problem 9.7-4 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection δ_{max}.



Solution 9.7-4 Simple beam with a rigid segment





FROM A TO B

$$v = -\frac{3\delta_B x}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$v' = -\frac{3\delta_B}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{2}$$

FROM B TO C

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

B.C. 1 At $x = L/3$, $v' = -\frac{3\delta_B}{L}$

$$\therefore C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \quad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \quad \left(\frac{L}{3} \le x \le L\right)$$

B.C. 2 $v(L) = 0$ $\quad \therefore C_2 = -\frac{PL^3}{54} + 3EI\delta_B$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^2}{54} + 3EI\delta_B \quad \left(\frac{L}{3} \le x \le L\right) \tag{5}$$

B.C. 3 At $x = \frac{L}{3}$, $(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$ (Eqs. 1 and 5)

$$\therefore \delta_B = \frac{8PL^3}{729EI} \quad \longleftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \longleftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI}(7L^3 - 61L^2x + 81Lx^2 - 27x^3) \quad \left(\frac{L}{3} \le x \le L\right) \tag{6}$$

Also,

$$v' = \frac{P}{486EI}(-61L^2 + 162Lx - 81x^2) \quad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

MAXIMUM DEFLECTION

$v' = 0$ gives $x_1 = \frac{L}{9}(9 - 2\sqrt{5}) = 0.5031L$

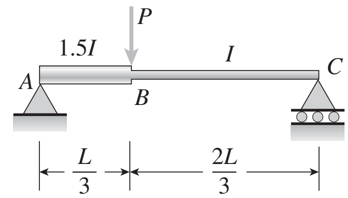
Substitute x_1 in Eq. (6) and simplify:

$$v_{max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{max} = -v_{max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363\frac{PL^3}{EI} \quad \longleftarrow$$

Problem 9.7-5 A simple beam ABC has moment of inertia $1.5I$ from A to B and I from B to C (see figure). A concentrated load P acts at point B.

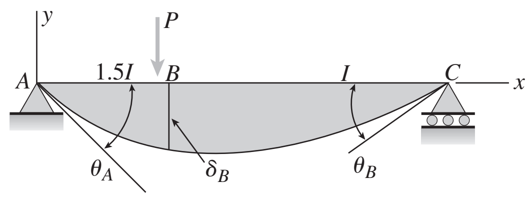
Obtain the equations of the deflection curves for both parts of the beam. From the equations, determine the angles of rotation θ_A and θ_C at the supports and the deflection δ_B at point B.



Solution 9.7-5 **Simple beam (nonprismatic)**

Use the bending-moment equation (Eq. 9-12a).

DEFLECTION CURVE



BENDING-MOMENT EQUATIONS

$$E\left(\frac{3I}{2}\right)v'' = M = \frac{2Px}{3} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \quad \left(\frac{L}{3} \le x \le L\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{4Px^2}{18} + C_1 \quad \left(0 \le x \le \frac{L}{3}\right) \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_2 \quad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

B.C. 1 Continuity of slopes at point B

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (4):

$$\frac{4P}{18}\left(\frac{L}{3}\right)^2 + C_1 = \frac{PL}{3}\left(\frac{L}{3}\right) - \frac{P}{6}\left(\frac{L}{3}\right)^2 + C_2$$

$$C_2 = C_1 - \frac{11PL^2}{162} \tag{5}$$

INTEGRATE EQS. (3) AND (4)

$$EIv = \frac{4Px^3}{54} + C_1x + C_3 \quad \left(0 \le x \le \frac{L}{3}\right) \tag{6}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} + C_2x + C_4 \quad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

B.C. 2 Deflection at support A

$$v(0) = 0 \quad \text{From Eq. (6):} \quad C_3 = 0 \tag{8}$$

B.C. 3 Deflection at support C

$$v(L) = 0 \quad \text{From Eq. (7):} \quad C_4 = -\frac{PL^3}{9} - C_2L \tag{9}$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$

From Eqs. (6), (8), and (7):

$$\frac{4P}{54}\left(\frac{L}{3}\right)^3 + C_1\left(\frac{L}{3}\right) = \frac{PL}{6}\left(\frac{L}{3}\right)^2 - \frac{P}{18}\left(\frac{L}{3}\right)^3 + C_2\left(\frac{L}{3}\right) + C_4$$

$$C_1L = \frac{10PL^3}{243} + C_2L + 3C_4 \tag{10}$$

SOLVE EQS (5), (8), (9), AND (10)

$$C_1 = -\frac{38PL^2}{729} \quad C_2 = -\frac{175PL^2}{1458} \quad C_3 = 0$$

$$C_4 = \frac{13PL^3}{1458}$$

SLOPES OF THE BEAM (FROM EQS. 3 AND 4)

$$v' = -\frac{2P}{729EI}(19L^2 - 81x^2) \quad \left(0 \le x \le \frac{L}{3}\right) \tag{11}$$

$$v' = -\frac{P}{1458EI}(175L^2 - 486Lx + 243x^2) \quad \left(\frac{L}{3} \le x \le L\right) \tag{12}$$

ANGLE OF ROTATION θ_A (FROM EQ. 11)

$$\theta_A = -v'(0) = \frac{38PL^2}{729EI} \quad \text{(positive clockwise)} \quad \longleftarrow$$

ANGLE OF ROTATION θ_C (FROM EQ. 12)

$$\theta_C = v'(L) = \frac{34PL^2}{729EI} \quad \text{(positive counterclockwise)} \quad \longleftarrow$$

DEFLECTIONS OF THE BEAM

Substitute C_1, C_2, C_3, and C_4 into Eqs. (6) and (7):

$$v = -\frac{2Px}{729EI}(19L^2 - 27x^2) \quad \left(0 \le x \le \frac{L}{3}\right) \quad \longleftarrow$$

$$v = -\frac{P}{1458EI}(-13L^3 + 175L^2x - 243Lx^2 + 81x^3) \quad \left(\frac{L}{3} \le x \le L\right) \quad \longleftarrow$$

DEFLECTION AT POINT B $\left(x = \frac{L}{3}\right)$

$$\delta_B = -v\left(\frac{L}{3}\right) = \frac{32PL^3}{2187\,EI} \quad \text{(positive downward)} \quad \longleftarrow$$

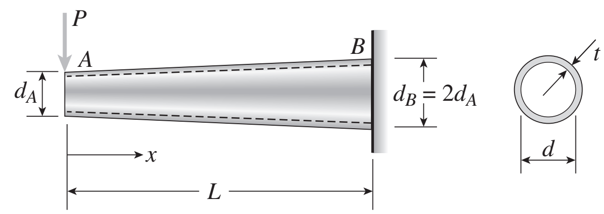
Problem 9.7-6 The tapered cantilever beam AB shown in the figure has thin-walled, hollow circular cross sections of constant thickness t. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi t d^3}{8} = \frac{\pi t d_A^3}{8L^3}(L + x)^3 = \frac{I_A}{L^3}(L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-6 **Tapered cantilever beam**

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{L^3}(L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^3}\right] \quad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{x\,dx}{(L + x)^3} = -\frac{L + 2x}{2(L + x)^2}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2\,(L + x)^2}\right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore\ C_1 = -\dfrac{3PL^2}{8EI_A}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2\,(L + x)^2}\right] - \frac{3PL^2}{8EI_A}$$

or

$$v' = \frac{PL^3}{EI_A}\left[\frac{L}{2\,(L + x)^2}\right] + \frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^2}\right] - \frac{3PL^2}{8EI_A} \quad (2)$$

INTEGRATE EQ. (2)

From Appendix C:

$$\int \frac{dx}{(L + x)^2} = -\frac{1}{L + x}$$

$$\int \frac{x\,dx}{(L + x)^2} = \frac{L}{L + x} + \ln(L + x)$$

$$v = \frac{PL^3}{EI_A}\left(\frac{L}{2}\right)\left(-\frac{1}{L + x}\right) + \frac{PL^3}{EI_A}\left[\frac{L}{L + x} + \ln(L + x)\right]$$

$$-\frac{3PL^2}{8EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)} + \ln(L + x) - \frac{3x}{8L}\right] + C_2 \quad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore\ C_2 = \dfrac{PL^3}{EI_A}\left[\dfrac{1}{8} - \ln(2L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)} - \frac{3x}{8L} + \frac{1}{8} + \ln\left(\frac{L + x}{2L}\right)\right] \quad \longleftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{8EI_A}(8\ln 2 - 5) = 0.06815\frac{PL^3}{EI_A}$$

(positive downward) $\longleftarrow$

NOTE: $\ln\dfrac{1}{2} = -\ln 2$

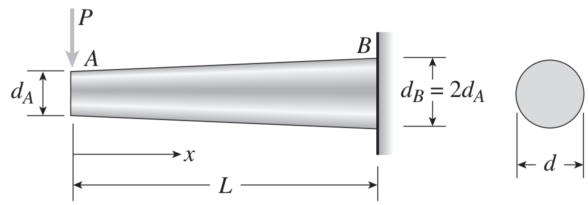
Problem 9.7-7 The tapered cantilever beam AB shown in the figure has a solid circular cross section. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi d^4}{64} = \frac{\pi d_A^4}{64L^4}(L + x)^4 = \frac{I_A}{L^4}(L + x)^4$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-7 Tapered cantilever beam

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{L^4}(L + x)^4$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^4}{EI_A}\left[\frac{x}{(L + x)^4}\right] \tag{1}$$

INTEGRATE EQ. (1)

From Appendix C: $$\int \frac{x dx}{(L + x)^4} = -\frac{L + 3x}{6(L + x)^3}$$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L + 3x}{6(L + x)^3}\right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\frac{PL^2}{12EI_A}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L + 3x}{6(L + x)^3}\right] - \frac{PL^2}{12EI_A}$$

or

$$v' = \frac{PL^4}{EI_A}\left[\frac{L}{6(L + x)^3}\right] + \frac{PL^4}{EI_A}\left[\frac{x}{2(L + x)^3}\right] - \frac{PL^2}{12EI_A} \tag{2}$$

INTEGRATE EQ. (2)

From Appendix C: $$\int \frac{dx}{(L + x)^3} = -\frac{1}{2(L + x)^2}$$

$$\int \frac{x dx}{(L + x)^3} = \frac{-(L + 2x)}{2(L + x)^2}$$

$$v = \frac{PL^4}{EI_A}\left(\frac{L}{6}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{L + x}\right)^2 + \frac{PL^4}{EI_A}\left(\frac{1}{2}\right)\left[-\frac{L + 2x}{2(L + x)^2}\right] - \frac{PL^2}{12EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[-\frac{L^2}{12(L + x)^2} - \frac{L(L + 2x)}{4(L + x)^2} - \frac{x}{12L}\right] + C_2 \tag{3}$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = \frac{PL^3}{EI_A}\left(\frac{7}{24}\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{24EI_A}\left[7 - \frac{4L(2L + 3x)}{(L + x)^2} - \frac{2x}{L}\right] \longleftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{24EI_A} \quad \text{(positive downward)} \longleftarrow$$

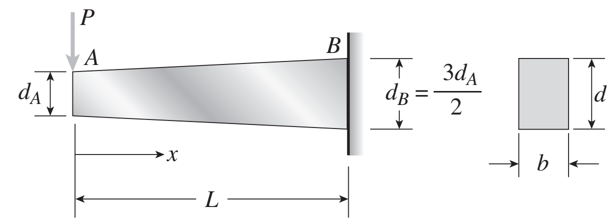
Problem 9.7-8 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular with constant width b, depth d_A at support A, and depth $d_B = 3d_A/2$ at the support. Thus, the depth d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{2L}(2L + x)$$

$$I = \frac{bd^3}{12} = \frac{bd_A^3}{96L^3}(2L + x)^3 = \frac{I_A}{8L^3}(2L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-8 **Tapered cantilever beam**

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{8L^3}(2L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{8PL^3}{EI_A}\left[\frac{x}{(2L + x)^3}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{xdx}{(2L + x)^3} = -\frac{2L + 2x}{2(2L + x)^2}$

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L + x}{(2L + x)^2}\right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\dfrac{16PL^2}{9EI_A}$

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L + x}{(2L + x)^2}\right] - \frac{16PL^2}{9EI_A}$$

or

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L}{(2L + x)^2}\right] + \frac{8PL^3}{EI_A}\left[\frac{x}{(2L + x)^2}\right] - \frac{16PL^2}{9EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $\displaystyle\int \frac{dx}{(2L + x)^2} = -\frac{1}{2L + x}$

$$\int \frac{xdx}{(2L + x)^2} = \frac{2L}{2L + x} + \ln(2L + x)$$

$$v = \frac{8PL^3}{EI_A}\left(-\frac{L}{2L + x}\right) + \frac{8PL^3}{EI_A}\left[\frac{2L}{2L + x} + \ln(2L + x)\right] - \frac{16PL^2}{9EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{8L}{2L + x} + 8\ln(2L + x) - \frac{16x}{9L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = -\dfrac{8PL^3}{EI_A}\left[\dfrac{1}{9} + \ln(3L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into EQ. (3).

$$v = \frac{8PL^3}{EI_A}\left[\frac{L}{2L + x} - \frac{2x}{9L} - \frac{1}{9} + \ln\left(\frac{2L + x}{3L}\right)\right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{8PL^2}{EI_A}\left[\ln\left(\frac{3}{2}\right) - \frac{7}{18}\right]$$

$$= 0.1326\frac{PL^3}{EI_A} \quad \text{(positive downward)} \quad \leftarrow$$

NOTE: $\ln\dfrac{2}{3} = -\ln\dfrac{3}{2}$

Problem 9.7-9 A simple beam ACB is constructed with square cross sections and a double taper (see figure). The depth of the beam at the supports is d_A and at the midpoint is $d_C = 2d_A$. Each half of the beam has length L. Thus, the depth d and moment of inertia I at distance x from the left-hand end are, respectively,

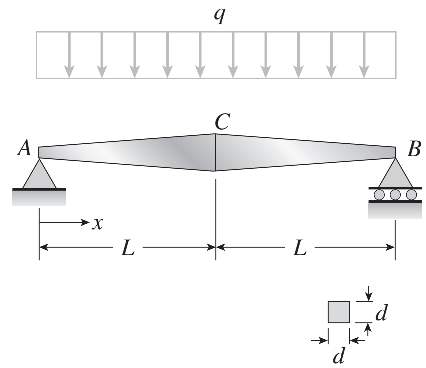
$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{d^4}{12} = \frac{d_A^4}{12L^4}(L + x)^4 = \frac{I_A}{L^4}(L + x)^4$$

in which I_A is the moment of inertia at end A of the beam. (These equations are valid for x between 0 and L, that is, for the left-hand half of the beam.)

(a) Obtain equations for the slope and deflection of the left-hand half of the beam due to the uniform load.

(b) From those equations obtain formulas for the angle of rotation θ_A at support A and the deflection δ_C at the midpoint.



Solution 9.7-9 Simple beam with a double taper

L = length of one-half of the beam

$$I = \frac{I_A}{L^4}(L + x)^4 \quad (0 \le x \le L)$$

(x is measured from the left-hand support A)

Reactions: $R_A = R_B = qL$

Bending moment: $M = R_A x - \frac{qx^2}{2} = qLx - \frac{qx^2}{2}$

From Eq. (9-12a):

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$v'' = \frac{qL^5x}{EI_A(L + x)^4} - \frac{qL^4x^2}{2EI_A(L + x)^4} \quad (0 \le x \le L) \quad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\int \frac{x\,dx}{(L + x)^4} = -\frac{L + 3x}{6(L + x)^3}$

$$\int \frac{x^2dx}{(L + x)^4} = -\frac{L^2 + 3Lx + 3x^2}{3(L + x)^3}$$

$$v' = \frac{qL^5}{EI_A}\left[-\frac{L + 3x}{6(L + x)^3}\right] - \frac{qL^4}{2EI_A}\left[-\frac{L^2 + 3Lx + 3x^2}{3(L + x)^3}\right] + C_1$$

$$= \frac{qL^4x^2}{2EI_A(L + x)^3} + C_1 \quad (0 \le x \le L) \quad (2)$$

B.C. 1 (symmetry) $v'(L) = 0 \quad \therefore C_1 = -\frac{qL^3}{16EI_A}$

SLOPE OF THE BEAM

Substitute C_1 into Eq. (2).

$$v' = \frac{qL^4x^2}{2EI_A(L + x)^3} - \frac{qL^3}{16EI_A}$$

$$= -\frac{qL^3}{16EI_A}\left[1 - \frac{8Lx^2}{(L + x)^3}\right] \quad (0 \le x \le L) \quad (3) \quad \leftarrow$$

ANGLE OF ROTATION AT SUPPORT A

$$\theta_A = -v'(0) = \frac{qL^3}{16EI_A} \quad \text{(positive clockwise)} \quad \leftarrow$$

INTEGRATE EQ. (3)

From Appendix C:

$$\int \frac{x^2dx}{(L + x)^3} = \frac{L(3L + 4x)}{2(L + x)^2} + \ln(L + x)$$

$$v = -\frac{qL^3}{16EI_A}\left[x - \frac{8L^2(3L + 4x)}{2(L + x)^2} - 8L\ln(L + x)\right] + C_2 \quad (0 \le x \le L) \quad (4)$$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = -\frac{qL^4}{2EI_A}\left(\frac{3}{2} + \ln L\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (4) and simplify. (The algebra is lengthy.)

$$v = -\frac{qL^4}{2EI_A}\left[\frac{(9L^2 + 14Lx + x^2)x}{8L(L + x)^2} - \ln\left(1 + \frac{x}{L}\right)\right]$$

$$(0 \le x \le L) \quad \leftarrow$$

DEFLECTION AT THE MIDPOINT C OF THE BEAM

$$\delta_C = -v(L) = \frac{qL^4}{8EI_A}(3 - 4\ln 2) = 0.02843\frac{qL^4}{EI_A}$$

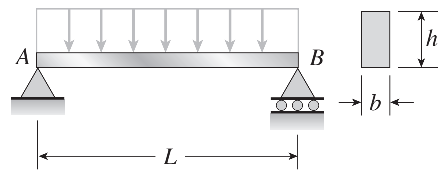
(positive downward) $\leftarrow$

Strain Energy

The beams described in the problems for Section 9.8 have constant flexural rigidity EI.

Problem 9.8-1 A uniformly loaded simple beam AB (see figure) of span length L and rectangular cross section (b = width, h = height) has a maximum bending stress σ_{max} due to the uniform load.

Determine the strain energy U stored in the beam.



Solution 9.8-1 Simple beam with a uniform load

Given: L, b, h, σ_{max} Find: U (strain energy)

Bending moment: $M = \frac{qLx}{2} - \frac{qx^2}{2}$

Strain energy (Eq. 9-80a): $U = \int_0^L \frac{M^2 dx}{2EI}$

$$= \frac{q^2L^5}{240EI} \quad (1)$$

Maximum stress: $\sigma_{max} = \frac{M_{max}c}{I} = \frac{M_{max}h}{2I}$

$M_{max} = \frac{qL^2}{8} \quad \sigma_{max} = \frac{qL^2h}{16I}$

Solve for q: $q = \frac{16I\sigma_{max}}{L^2h}$

Substitute q into Eq. (1):

$$U = \frac{16I\sigma_{max}^2 L}{15h^2E}$$

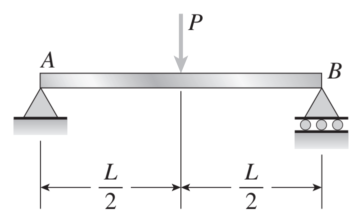
Substitute $I = \frac{bh^3}{12}$: $U = \frac{4bhL\sigma_{max}^2}{45E}$ ←

Problem 9.8-2 A simple beam AB of length L supports a concentrated load P at the midpoint (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.

(b) Evaluate the strain energy of the beam from the equation of the deflection curve.

(c) From the strain energy, determine the deflection δ under the load P.



Solution 9.8-2 Simple beam with a concentrated load

(a) Bending moment $M = \frac{Px}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$

Strain energy (Eq. 9-80a): $U = 2\int_0^{L/2} \frac{M^2 dx}{2EI} = \frac{P^2L^3}{96EI}$ ←

(b) Deflection curve

From Table G-2, Case 4:

$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \quad \left(0 \le x \le \frac{L}{2}\right)$

$\frac{dv}{dx} = -\frac{P}{16EI}(L^2 - 4x^2) \quad \frac{d^2v}{dx^2} = \frac{Px}{2EI}$

Strain energy (Eq. 9-80b):

$$U = 2\int_0^{L/2} \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = EI\int_0^{L/2}\left(\frac{Px}{2EI}\right)^2 dx$$

$$= \frac{P^2L^3}{96EI} \quad ←$$

(c) Deflection δ under the load P

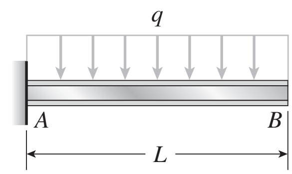
From Eq. (9-82a):

$\delta = \frac{2U}{P} = \frac{PL^3}{48EI}$ ←

Problem 9.8-3 A cantilever beam *AB* of length *L* supports a uniform load of intensity *q* (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.

(b) Evaluate the strain energy of the beam from the equation of the deflection curve.



Solution 9.8-3 Cantilever beam with a uniform load

(a) BENDING MOMENT

Measure *x* from the free end *B*.

$$M = -\frac{qx^2}{2}$$

Strain energy (Eq. 9-80a):

$$U = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \left(\frac{1}{2EI}\right)\left(-\frac{qx^2}{2}\right)^2 dx = \frac{q^2L^5}{40EI} \longleftarrow$$

(b) DEFLECTION CURVE

Measure *x* from the fixed support *A*.
From Table G-1, Case 1:

$$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2)$$

$$\frac{dv}{dx} = -\frac{q}{6EI}(3L^2x - 3Lx^2 + x^3)$$

$$\frac{d^2v}{dx^2} = -\frac{q}{2EI}(L^2 - 2Lx + x^2)$$

Strain energy (Eq. 9-80b):

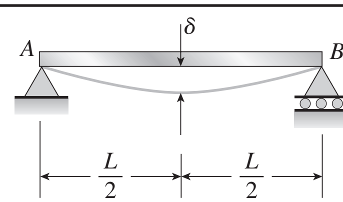
$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx$$

$$= \frac{EI}{2}\int_0^L \left(-\frac{q}{2EI}\right)^2 (L^2 - 2Lx + x^2)^2 dx$$

$$= \frac{q^2L^5}{40EI} \longleftarrow$$

Problem 9.8-4 A simple beam *AB* of length *L* is subjected to loads that produce a symmetric deflection curve with maximum deflection δ at the midpoint of the span (see figure).

How much strain energy *U* is stored in the beam if the deflection curve is (a) a parabola, and (b) a half wave of a sine curve?



Solution 9.8-4 Simple beam (symmetric deflection curve)

GIVEN: *L*, *EI*, δ   δ = maximum deflection at midpoint

Determine the strain energy *U*.
Assume the deflection *v* is positive downward.

(a) DEFLECTION CURVE IS A PARABOLA

$$v = \frac{4\delta x}{L^2}(L - x) \qquad \frac{dv}{dx} = \frac{4\delta}{L^2}(L - 2x)$$

$$\frac{d^2v}{dx^2} = -\frac{8\delta}{L^2}$$

Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(-\frac{8\delta}{L^2}\right)^2 dx = \frac{32EI\delta^2}{L^3} \longleftarrow$$

(b) DEFLECTION CURVE IS A SINE CURVE

$$v = \delta \sin\frac{\pi x}{L} \qquad \frac{dv}{dx} = \frac{\pi\delta}{L}\cos\frac{\pi x}{L} \qquad \frac{d^2v}{dx^2} = -\frac{\pi^2\delta}{L^2}\sin\frac{\pi x}{L}$$

Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(-\frac{\pi^2\delta}{L^2}\right)^2 \sin^2\frac{\pi x}{L}\,dx$$

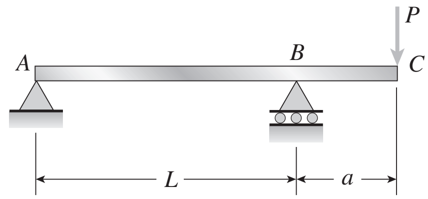
$$= \frac{\pi^4 EI\delta^2}{4L^3} \longleftarrow$$

Problem 9.8-5 A beam ABC with simple supports at A and B and an overhang BC supports a concentrated load P at the free end C (see figure).

(a) Determine the strain energy U stored in the beam due to the load P.

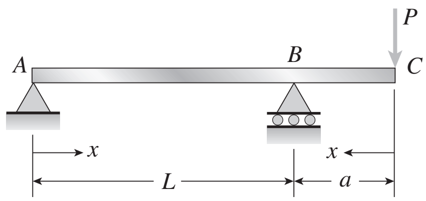
(b) From the strain energy, find the deflection δ_C under the load P.

(c) Calculate the numerical values of U and δ_C if the length L is 8 ft, the overhang length a is 3 ft, the beam is a W 10 × 12 steel wide-flange section, and the load P produces a maximum stress of 12,000 psi in the beam. (Use $E = 29 \times 10^6$ psi.)



Solution 9.8-5 Simple beam with an overhang

(a) STRAIN ENERGY (use Eq.9-80a)



FROM A TO B: $M = -\dfrac{Pax}{L}$

$$U_{AB} = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{1}{2EI}\left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

FROM B TO C: $M = -Px$

$$U_{BC} = \int_0^a \frac{1}{2EI}(-Px)^2 dx = \frac{P^2a^3}{6EI}$$

TOTAL STRAIN ENERGY:

$$U = U_{AB} + U_{BC} = \frac{P^2a^2}{6EI}(L + a) \quad \longleftarrow$$

(b) DEFLECTION δ_C UNDER THE LOAD P

From Eq. (9-82a):

$$\delta_C = \frac{2U}{P} = \frac{Pa^2}{3EI}(L + a) \quad \longleftarrow$$

(c) CALCULATE U AND δ_c

Data: $L = 8$ ft $= 96$ in. $a = 3$ ft $= 36$ in.

W 10 × 12 $E = 29 \times 10^6$ psi

$\sigma_{max} = 12{,}000$ psi

$I = 53.8\text{ in.}^4 \quad c = \dfrac{d}{2} = \dfrac{9.87}{2} = 4.935$ in.

Express load P in terms of maximum stress:

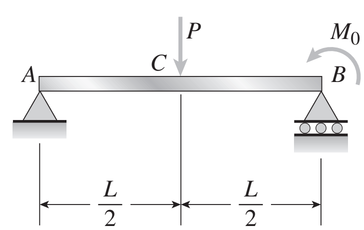
$$\sigma_{max} = \frac{Mc}{I} = \frac{M_{max}c}{I} = \frac{Pac}{I} \qquad \therefore\ P = \frac{\sigma_{max} I}{ac}$$

$$U = \frac{P^2a^2(L + a)}{6EI} = \frac{\sigma_{max}^2 I(L + a)}{6c^2E} = 241\text{ in.-lb} \quad \longleftarrow$$

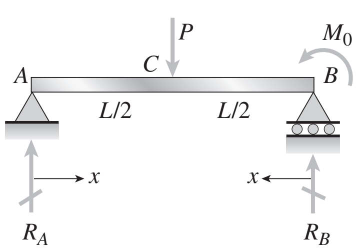
$$\delta_C = \frac{Pa^2(L + a)}{3EI} = \frac{\sigma_{max} a(L + a)}{3cE} = 0.133\text{ in.} \quad \longleftarrow$$

Problem 9.8-6 A simple beam ACB supporting a concentrated load P at the midpoint and a couple of moment M_0 at one end is shown in the figure.

Determine the strain energy U stored in the beam due to the force P and the couple M_0 acting simultaneously.



Solution 9.8-6 Simple beam with two loads



$$R_A = \frac{P}{2} + \frac{M_0}{L}$$

$$R_B = \frac{P}{2} - \frac{M_0}{L}$$

FROM A TO C $\quad M = R_A x = \left(\dfrac{P}{2} + \dfrac{M_0}{L}\right)x$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^{L/2}\left(\frac{P}{2} + \frac{M_0}{L}\right)^2 x^2 dx$$

$$= \frac{L}{192EI}(P^2L^2 + 4PLM_0 + 4M_0^2)$$

(Continued)

FROM B TO C $\quad M = R_B x + M_0 = \left(\frac{P}{2} - \frac{M_0}{L}\right)x + M_0$

$$U_{BC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^{L/2}\left[\left(\frac{P}{2} - \frac{M_0}{L}\right)x + M_0\right]^2 dx$$

$$= \frac{L}{192EI}(P^2L^2 + 8PLM_0 + 28M_0^2)$$

STRAIN ENERGY OF THE ENTIRE BEAM

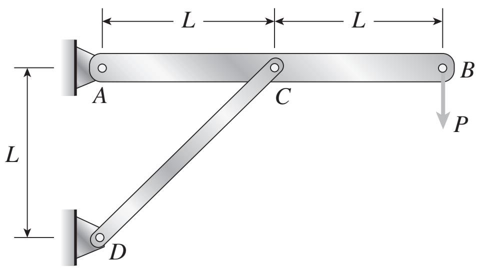
$$U = U_{AC} + U_{BC} = \frac{L}{96EI}(P^2L^2 + 6PLM_0 + 16M_0^2)$$

$$= \frac{P^2L^3}{96EI} + \frac{PM_0L^2}{16EI} + \frac{M_0^2 L}{6EI} \quad \longleftarrow$$

Problem 9.8-7 The frame shown in the figure consists of a beam ACB supported by a strut CD. The beam has length $2L$ and is continuous through joint C. A concentrated load P acts at the free end B.

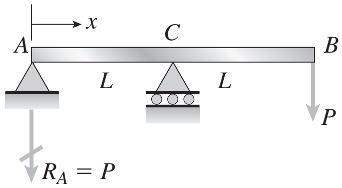
Determine the vertical deflection δ_B at point B due to the load P.

Note: Let EI denote the flexural rigidity of the beam, and let EA denote the axial rigidity of the strut. Disregard axial and shearing effects in the beam, and disregard any bending effects in the strut.



Solution 9.8-7 Frame with beam and strut

BEAM ACB



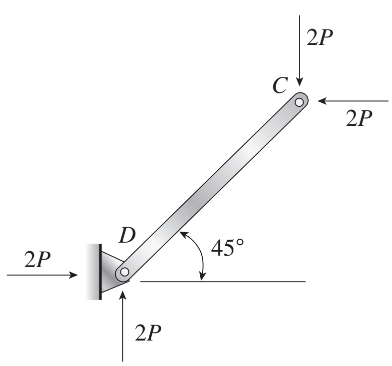
For part AC of the beam: $M = -Px$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px)^2 dx = \frac{P^2L^3}{6EI}$$

For part CB of the beam: $U_{CB} = U_{AC} = \dfrac{P^2L^3}{6EI}$

Entire beam: $U_{\text{BEAM}} = U_{AC} + U_{CB} = \dfrac{P^2L^3}{3EI}$

STRUT CD



L_{CD} = length of strut
$= \sqrt{2}L$

F = axial force in strut
$= 2\sqrt{2}P$

$$U_{\text{STRUT}} = \frac{F^2 L_{CD}}{2EA} \quad \text{(Eq. 2-37a)}$$

$$U_{\text{STRUT}} = \frac{(2\sqrt{2}P)^2(\sqrt{2}L)}{2EA} = \frac{4\sqrt{2}P^2L}{EA}$$

FRAME $\quad U = U_{\text{BEAM}} + U_{\text{STRUT}} = \dfrac{P^2L^3}{3EI} + \dfrac{4\sqrt{2}P^2L}{EA}$

DEFLECTION δ_B AT POINT B

From Eq. (9-82 a):

$$\delta_B = \frac{2U}{P} = \frac{2PL^3}{3EI} + \frac{8\sqrt{2}PL}{EA} \quad \longleftarrow$$